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An Algebraic Treatment of Diophantine Analysis

Anthony Andrew Aucoin

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AN ALGEBRAIC TREATMENT OF DIOPHANTINE ANALYSIS

A Dissertation

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Without the help and
encouragement of Professor
W. V. Parker this dissertation
would not have been written.

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ABSTRACT

Diophantine analysis, while an old subject, is a collection of special problems. The methods used are very varied and those used to solve one problem will not usually solve a second one. It is possible that this condition exists because Diophantine analysis is usually considered as number theory. While some problems are problems in number theory, a large body of material may be developed by algebraic methods. In this thesis the methods of algebra are exploited.

The material of this thesis divides itself into chapters naturally, according to the degree of the equations. Considered in order are quadratics, cubics, quartics, and equations of degree n. All equations considered are homogeneous polynomials except a few of degree n. All except two equations are generalized to equations of degree n. The solutions are given in terms of arbitrary parameters and are integral for an integral choice of the parameters.

An important concept of this thesis is the introduction of equivalent solutions (due to Professor W. V. Parker). Consider the Diophantine equation

$$(1) \qquad f(x_1, \ldots, x_p) = g(y_1, \ldots, y_q),$$

where f and g are homogeneous polynomials with integral coefficients, of degrees m and n respectively. If $x_i = \alpha_i$, $y_j = \beta_j$ is a solution of (1) and there are no integers $s > 1$, α_i', β_j' such that $\alpha_i = \alpha_i' s^\lambda$, $\beta_j = \beta_j' s^\mu$ where λ, μ are relatively prime positive integers such that $\lambda m = \mu n$, then

$x_i = \alpha_i$, $y_j = \beta_j$ is defined to be a primitive solution of (1). If $x_i = \alpha_i$, $y_j = \beta_j$ is a primitive solution of (1), then $x_i = \alpha_i t^\lambda$, $y_j = \beta_j t^\mu$, where t is a non-zero integer, is also a solution which is said to be derived from this primitive solution. Two solutions are defined to be equivalent if they may be derived from the same primitive solution.

Each solution obtained is shown, within certain limits, to be general. For a given solution of one of the equations considered, the parameters are chosen in such a way that a solution, equivalent to the given one, is obtained, provided the given solution is not also a solution of a certain equation. Usually, if the original equation is of degree n, the "certain" equation is of degree n - 1.

CHAPTER I

INTRODUCTION

Diophantine analysis treats the problem of finding rational numbers or integers which satisfy the equation

$$f(x_1, \ldots, x_p) = 0.$$

Usually f is a polynomial with integral coefficients. Few problems have been solved when f is not a polynomial. The solutions are given in terms of arbitrary parameters.

This thesis will be concerned with finding, in terms of parameters, solutions which will be integral for an integral choice of the parameters. The functions will be polynomials except in several problems considered at the end of Chapter V.

While Diophantine analysis is an old subject and has been studied extensively, it remains for the most part a collection of special problems. It appears that this is due to the subject's being unfortunately linked with number theory rather than with algebra. It is true that some problems in Diophantine analysis are problems in number theory, but a large body of material may be developed by strictly algebraic methods. As an insight into the state of development of the subject we quote from Carmichael:[1] "When we pass to equations

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1/ Robert D. Carmichael, Diophantine Analysis (New York: John Wiley and Sons, Inc., 1915), p. 85.

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of degree higher than the fourth we find but little effective progress has been made. Often it is a matter of great difficulty to determine whether a given solution is the general solution of a given equation. Indeed this is true, to a large extent, of equations of the third and fourth degrees. Even here there are but few equations for which a general solution is known or for which it is known that no solution at all exists. As the degree of the equation increases, the generality of the known results decreases in a rapid ratio. Only the most special equations of degree higher than the fourth have been at all treated and for only a few of these is one able to answer the question naturally propounded as to the existence or generality of solutions."

If $x_i = \alpha_i$, $y_j = \beta_j$ is a solution of the equation

(1:1) $$f(x_1, \ldots, x_p) = g(y_1, \ldots, y_q)$$

where f and g are homogeneous polynomials of degrees m and n, respectively, and there are no integers $s > 1$, α_i', β_j' such that $\alpha_i = \alpha_i' s^\lambda$, $\beta_j = \beta_j' s^\mu$, where λ, μ are relatively prime positive integers such that $\lambda m = \mu n$, then $x_i = \alpha_i$, $y_j = \beta_j$ is defined to be a primitive solution of the equation. If $x_i = \alpha_i$, $y_j = \beta_j$ is a primitive solution of (1:1), then $x_i = \alpha_i t^\lambda$, $y_j = \beta_j t^\mu$, where t is a non-zero integer, is also a solution, which is said to be <u>derived</u> from this primitive solution. Two solutions are defined to be equivalent if they may be derived from the same primitive solution.[2/]

2/ This definition of equivalence of solution as well as it's subsequent application is due to Professor W. V. Parker.

A primitive solution of a Diophantine equation has been defined as a set of relatively prime integers which satisfy the equation.[3/] Under this definition some equations, for

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3/ Robert D. Carmichael, *op. cit.*, p. 2.

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example, $x^4 - y^4 = z^3$, have solutions but do not have primitive solutions. However, $x = \alpha(\alpha^4 - \beta^4)^2$, $y = \beta(\alpha^4 - \beta^4)^2$, $z = (\alpha^4 - \beta^4)^3$ is a primitive solution of this equation according to the definition of this thesis. From a relative prime (primitive) solution other solutions may be obtained by multiplying each by a power of some constant. In this thesis, a primitive solution is obtained by dividing any solution by a power of some constant. It is believed that one procedure is as general as the other.

CHAPTER II

QUADRATIC EQUATIONS

A. Desboves[4/] has shown that, given a solution of the

4/ Leonard E. Dickson, History of the Theory of Numbers (Washington: Carnegie Institution of Washington, 1920), vol. 2, p. 432.

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the equation $\sum_{i,j}^{n} a_{ij} x_i x_j = 0$, other solutions may be determined. The method is shown in the first theorem.

Theorem 2:1. If $x_i = \alpha_i$ is a solution of the equation

(2:1) $$\sum_{i,j}^{n} a_{ij} x_i x_j = 0,$$

then every solution, which is not also a solution of

(2:2) $$\sum_{i,j}^{n} a_{ij} \alpha_i x_j = 0,$$

is equivalent to one of the solutions given by

(2:3) $$x_i = \alpha_i s - \lambda_i t$$

where

(2:4) $$s = \sum_{i,j}^{n} a_{ij} \lambda_i \lambda_j \quad , \qquad t = 2 \sum_{i,j}^{n} a_{ij} \alpha_i \lambda_j$$

and the λ's are arbitrary integers.

Proof. If x_i have the values given by (2:3), then (2:1) becomes $2st \sum_{i,j}^{n} a_{ij} \alpha_i \lambda_j = t^2 \sum_{i,j}^{n} a_{ij} \lambda_i \lambda_j$, which is satisfied identically in the λ's if s and t are given by (2:4).

Suppose now that $x_i = \rho_i$ is a given solution of (2:1). If we choose $\lambda_i = \rho_i$, then s = 0 and (2:3) becomes $x_i = -\rho_i t$ which is equivalent to the given solution provided $t \neq 0$; that is, provided $x_i = \rho_i$ is not also a solution of (2:2).

Desboves states that (2:3) gives all solutions of (2:1) if two equivalent solutions are regarded as the same solution. That this is not the case is seen from the proof of Theorem 2:1.

Various quadratic equations will now be solved without considering a particular solution. Each equation considered is such, however, that a particular solution may be noted by inspection.

Theorem 2:2. Every solution of the equation

(2:5) $$\sum_i^n a_i^2 x_i^2 = \sum_i^n b_i^2 z^2,$$

where all the a's are different from zero and $\sum_i^n b_i^2 \neq 0$, is equivalent to one of the solutions given by

(2:6) $$x_j = 2K\alpha_j \sum_i^n a_i b_i \alpha_i - K_j b_j \sum_i^n a_i^2 \alpha_i^2, \quad (j = 1,..,n),$$
$$z = K\sum_i^n a_i^2 \alpha_i^2,$$

where K is the least common multiple of the a's, $K_j = K/a_j$ and the α's are arbitrary integers.

Proof. Write (2:5) in the form

(2:7) $$\sum_i^n (b_i^2 z^2 - a_i^2 x_i^2) = 0.$$

If

(2:8) $$b_j z + a_j x_j = a_j \alpha_j t, \quad (j = 1,..,n)$$

then (2:7) becomes

(2:9) $$2zt \sum_i^n a_i b_i \alpha_i = t^2 \sum_i^n a_i^2 \alpha_i^2,$$

which is satisfied identically in the α's if $z = K\sum_i^n a_i^2 \alpha_i^2$, $t = 2K\sum_i^n a_i b_i \alpha_i$. Hence a solution of (2:5) is given by (2:6).

Suppose now that $x_j = \lambda_j$, $z = \rho$ is any solution of (2:5). Then

(2:10) $$\sum_i^n a_i^2 \lambda_i^2 = \sum_i^n b_i^2 \rho^2.$$

If

(2:11) $$\alpha_j = K\lambda_j + K_j b_j \rho, \quad (j = 1,..,n)$$

then (2:6) becomes

(2:12) $$z = \rho R, \; x_j = \lambda_j R, \quad (j = 1,..,n)$$

where $R = 2K^2 \sum_i^n b_i(a_i\lambda_i + b_i\rho)$. However if

(2:13) $\alpha_j = K\lambda_j - K_j b_j \rho$, $(j = 1,..,n)$

then (2:6) becomes

(2:14) $z = -\rho S,\ x_j = \lambda_j S,$ $(j = 1,..,n)$

where $S = 2K^3 \sum_i^n b_i(a_i\lambda_i - b_i\rho)$.

Now if either R or S is non-zero, then either (2:12) or (2:14) is a solution of (2:5), each of which is equivalent to the solution $z = \rho$, $x_j = \lambda_j$.

It will now be shown that R and S are not both zero unless the solution is the trivial one $\rho = \lambda_j = 0$. If $R = 0$, then since $K \neq 0$, $\sum_i^n b_i(a_i\lambda_i + b_i\rho) = 0$, and hence

(2:15) $\sum_i^n b_i\rho(a_i\lambda_i + b_i\rho) = 0.$

From (2:10), (2:15) becomes

(2:16) $\sum_i^n a_i\lambda_i(a_i\lambda_i + b_i\rho) = 0.$

By adding (2:15) and (2:16) it follows that $\sum_i^n (a_i\lambda_i + b_i\rho)^2 = 0$, and hence

(2:17) $a_j\lambda_j + b_j\rho = 0,$ $(j = 1,..,n)$.

Similarly, if $S = 0$,

(2:18) $a_j\lambda_j - b_j\rho = 0,$ $(j = 1,..,n)$.

Since all the a's are different from zero, each λ_j must be zero in order that (2:17) and (2:18) be true simultaneously, and hence $\rho = 0$.

If each $a_i = 1$, $b_1 = 1$, $b_i = 0$ $(i = 2,3,..,n)$, (2:5) reduces to the sum of n squares equal to a square. The solution in this case is given by 5/ $z = \sum_i^n \alpha_i^2$, $x_1 = \alpha_1^2 - \sum_{i=2}^n \alpha_i^2$,

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5/ Leonard E. Dickson, op. cit., p.318.

$x_j = 2\alpha_1\alpha_j$, $(j = 2,3,..,n)$. In particular, for $n = 3$, the solutions of

(2:19) $$x^2 + y^2 + z^2 = t^2$$

are

(2:20)
$$t = \alpha^2 + \beta^2 + \gamma^2 \quad , \quad x = \alpha^2 - \beta^2 - \gamma^2$$
$$y = 2\alpha\beta \quad , \quad z = 2\alpha\gamma .$$

Catalan 6/ has pointed out that these formulas do not

6/ Leonard E. Dickson, op. cit., p. 269.

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give all solutions of (2:19), but that all relatively prime solutions are given by

(2:21)
$$t = m^2 + n^2 + p^2 + q^2 \quad , \quad x = m^2 - n^2 - p^2 + q^2$$
$$y = 2mn + 2pq \quad , \quad z = 2mp - 2nq.$$

If in (2:20) the choice $\alpha = m^2 + q^2$, $\beta = mn + pq$, $\gamma = mp - nq$ is made, then

$$t = (m^2+q^2)(m^2 + n^2 + p^2 + q^2),$$
$$x = (m^2+q^2)(m^2 - n^2 - p^2 + q^2),$$
$$y = (m^2+q^2)(2mn + 2pq),$$
$$z = (m^2+q^2)(2mp - 2nq),$$

which is equivalent to (2:21) provided $m^2 + q^2 \neq 0$. If $m = q = 0$, then (2:21) becomes $t = n^2 + p^2$, $x = - n^2 - p^2$, $y = z = 0$, which is given by (2:20) with $\alpha = 0$, $\beta = n$, $\gamma = p$. Hence in every case, for any solution given by (2:21), an equivalent solution is given by (2:20).

The solutions of the equation

(2:22) $$\sum_i^n d_i x_i^2 = \sum_i^n d_i z^2$$

may be determined by Theorem 2:2 by setting $a_i^2 = b_i^2 = d_i$. After removing the common factor K from each expression in (2:6)

(2:23) $$z = \sum_i^n d_i \alpha_i^2 \ , \ x_j = 2\alpha_j \sum_i^n d_i \alpha_i - \sum_i^n d_i \alpha_i^2 \ , \ (j = 1, \ldots, n)$$

follows as a solution of (2:22). Since (2:23) satisfies (2:22) identically in the α's, it is not necessary to assume that d_i be a square or that it be positive. It may be shown as above, that (2:23) gives a solution equivalent to any existing solution of (2:22).

Theorem 2:3. The equation

(2:24) $$\sum_i^n (a_i x_i + b_i y_i)(c_i x_i + d_i y_i) = 0,$$

has solutions and every solution which is not also a solution of

(2:25) $$\sum_i^n a_i c_i (K x_i + b_i K_i y_i)^2 = 0,$$

where K is the least common multiple of the a's, $K_j = K/a_j$, is equivalent to one of the solutions given by

(2:26) $$x_i = \alpha_i t - b_i K_i \beta_i s, \qquad y_i = \beta_i Ks$$

where

(2:27) $$s = \sum_i^n a_i c_i \alpha_i^2, \qquad t = K \sum_i^n \alpha_i \beta_i (b_i c_i - a_i d_i),$$

and the α's and β's are arbitrary integers.

Proof. Let $a_i x_i + b_i y_i = a \alpha_i t$, $y_i = \beta_i Ks$. Then $x_i = \alpha_i t - b_i K_i \beta_i s$. Substituting in (2:24) gives

$$t^2 \sum_i^n a_i c_i \alpha_i^2 = stK \sum_i^n \alpha_i \beta_i (b_i c_i - a_i d_i)$$

which is identically satisfied in the α's and β's if s and t are given by (2:27). Hence (2:26) is a solution of (2:24).

Suppose $x_i = \lambda_i$, $y_i = \mu_i$ is any given solution of (2:24). If $\alpha_i = K\lambda_i + b_i K_i \mu_i$, $\beta_i = \mu_i$, then

$$t = K \sum_i^n \mu_i (b_i c_i - a_i d_i)(K\lambda_i + b_i K_i \mu_i),$$

$$s = \sum_i^n a_i c_i (K\lambda_i + b_i K_i \mu_i)^2$$

$$= K^2 \sum_i^n (a_i c_i \lambda_i^2 + b_i c_i \lambda_i \mu_i) + K^2 \sum_i^n b_i c_i \lambda_i \mu_i + \sum_i^n a_i b_i^2 c_i K_i^2 \mu_i^2,$$

and since $\sum_i^n (a_i c_i \lambda_i^2 + b_i c_i \lambda_i \mu_i) = - \sum_i^n (a_i d_i \lambda_i \mu_i + b_i d_i \mu_i^2)$,

$$\begin{aligned} s &= \sum_i^n (-K^2 a_i d_i \lambda_i \mu_i - K^2 b_i d_i \mu_i^2 + K^2 b_i c_i \lambda_i \mu_i + a_i b_i^2 c_i K_i \mu_i^2) \\ &= K \sum_i^n \mu_i (b_i c_i - a_i d_i)(K \lambda_i + b_i K_i \mu_i) \\ &= t. \end{aligned}$$

Hence $x_i = \lambda_i Ks$, $y_i = \mu_i Ks$ which is equivalent to the given solution if $x_i = \lambda_i$, $y_i = \mu_i$ is not also a solution of (2:25).

It will be noted that this also solves the equation $\sum_i^n (a_i x_i + b_i y)(c_i x_i + d_i y) = 0$, of which (2:5) is a special case. In this case let $\beta_i = 1$ in (2:26).

Likewise the general homogeneous quadratic equation may be solved if it contains either two or three terms which are quadratic in two unknowns, and which are factorable into the product of two linear factors with integral coefficients. This is shown in the next theorem.

Theorem 2:4. The equation

$$(2{:}28) \qquad (ay + bz)(cy + dz) = y \sum_i^n a_i x_i + z \sum_i^n b_i x_i + \sum_{i,j}^n c_{ij} x_i x_j,$$

where a and b are not both zero, and c and d are not both zero, has solutions, and every solution which is not a solution of

$$(2{:}29) \qquad (bc - ad)(ay + bz) + \sum_i^n (ab_i - ba_i) x_i = 0,$$

is equivalent to one of the solutions given by

$$(2{:}30) \qquad \begin{aligned} y &= -d\alpha^2 + \alpha \sum_i^n b_i \alpha_i + b \sum_{i,j}^n c_{ij} \alpha_i \alpha_j, \\ z &= c\alpha^2 - \alpha \sum_i^n a_i \alpha_i - a \sum_{i,j}^n c_{ij} \alpha_i \alpha_j, \\ x_k &= \alpha_k \Big[(bc - ad)\alpha + \sum_i^n (ab_i - ba_i)\alpha_i \Big], \quad (k = 1, \ldots, n), \end{aligned}$$

where α, α_i are arbitrary integers.

Proof. Let

$$(2{:}31) \qquad ay + bz = \alpha t, \qquad x_k = \alpha_k t,$$

then (2:28) becomes

$$(2{:}32)\qquad \alpha t(cy+dz) = yt\sum_i^n a_i\alpha_i + zt\sum_i^n b_i\alpha_i + t^2\sum_{i,j}^n c_{ij}\alpha_i\alpha_j .$$

The combination of (2:31) and (2:32) gives

$$[(bc - ad)\alpha + \sum_i^n(ab_i - ba_i)\alpha_i]\,ty = [-d\alpha^2 + \alpha\sum_i^n b_i\alpha_i + b\sum_{i,j}^n c_{ij}\alpha_i\alpha_j]t^2,$$
$$[(bc - ad)\alpha + \sum_i^n(ab_i - ba_i)\alpha_i]\,tz = [c\alpha^2 - \alpha\sum_i^n a_i\alpha_i - a\sum_{i,j}^n c_{ij}\alpha_i\alpha_j]t^2$$

which are satisfied identically in the α's if

$$y = -d\alpha^2 + \alpha\sum_i^n b_i\alpha_i + b\sum_{i,j}^n c_{ij}\alpha_i\alpha_j ,$$
$$z = c\alpha^2 - \alpha\sum_i^n a_i\alpha_i - a\sum_{i,j}^n c_{ij}\alpha_i\alpha_j ,$$
$$t = (bc - ad)\alpha + \sum_i^n(ab_i - ba_i)\alpha_i .$$

Hence (2:30) is a solution of (2:28).

Suppose now that $y=\rho$, $z=\sigma$, $x_k=\lambda_k$ is any solution of (2:28). If in (2:30), $\alpha = a\rho + b\sigma$, $\alpha_k = \lambda_k$, then $y=\rho P$, $z=\sigma P$, $x_k=\lambda_k P$ where

$$P = (bc - ad)(a\rho + b\sigma) + \sum_i^n(ab_i - ba_i)\lambda_i ,$$

which is equivalent to the given solution provided $P \neq 0$; that is, provided $y=\rho$, $z=\sigma$, $x_k=\lambda_k$ is not a solution of (2:29).

Interchanging a with c, b with d in (2:30) gives

$$y = -b\alpha^2 + \alpha\sum_i^n b_i\alpha_i + d\sum_{i,j}^n c_{ij}\alpha_i\alpha_j ,$$
$$(2{:}33)\qquad z = a\alpha^2 - \alpha\sum_i^n a_i\alpha_i - c\sum_{i,j}^n c_{ij}\alpha_i\alpha_j ,$$
$$x_k = \alpha_k[(ad - bc)\alpha + \sum_i^n(cb_i - da_i)\alpha_i],\quad (k=1,\ldots,n),$$

a solution equivalent to any existing solution which does not satisfy

$$(2{:}34)\qquad (ad - bc)(cy + dz) + \sum_i^n(cb_i - da_i)x_i = 0.$$

A solution of the sum of m squares equal to the sum of n squares may be obtained by Theorem 2:4 or by Theorem 2:2 after a transformation of the equation. The equation

$$(2{:}35)\qquad \sum_i^m x_i^2 = \sum_1^n y_i^2 \qquad m \geq n,$$

may be written in the form

$$(x_1 - y_1)(x_1 + y_1) = \sum_{j=2}^{n} y_j^2 - \sum_{i=2}^{m} x_i^2 ,$$

which is a special case of Theorem 2:4. However, letting $y_j = \alpha_j t$ $(j = 1,..,n)$, (2:35) becomes

(2:36) $$\sum_{i}^{m} x_i^2 = \sum_{j}^{n} \alpha_j^2 t^2$$

where $\alpha_j = 0$ for $j > n$. Then (2:36) is a special case of Theorem 2:2, and may be solved for $x_1, x_2, \ldots, x_m, t$ in terms of m parameters. It will be noted that the first method gives the solution in terms of $m + n - 1$, while the second method gives the solution in terms of $m + n$ parameters.

A combination of Theorem 2:1 and Theorem 2:4 may be used to obtain a solution equivalent to any solution of (2:28). For example, consider the equation

(2:37) $$2x^2 + 3xy + y^2 - xz - yz + yw - 2zw = 0,$$

which may be written in the form

$$(x + y)(2x + y) = xz + yz - yw + 2zw.$$

From (2:30) and (2:33) the two solutions

(2:38)
$$\begin{aligned} x &= -p^2 + pq - pr + 2qr, \\ y &= 2p^2 - pq - 2qr, \\ z &= pq - qr, \\ w &= pr - r^2, \end{aligned}$$

and

(2:39)
$$\begin{aligned} x &= -p^2 + pq - pr + 2qr, \\ y &= p^2 - pq - 4qr, \\ z &= -pq + q^2 - 2qr, \\ w &= -pr + qr - 2r^2, \end{aligned}$$

are obtained.

Any solution which does not satisfy

(2:40) $$x + y - w = 0$$

is equivalent to one given by (2:38), and any solution which does not satisfy

(2:41) $$2x + y - z + 2w = 0$$

is equivalent to one given by (2:39).

Now (2:37), (2:40), and (2:41) have the common solutions (1, -1, 1, 0) and (-7, 10, 2, 3). (2, -2, -1, -1) is a particular solution of (2:37) obtained from (2:38) by choosing $p = 0$, $q = r = 1$. From this particular solution Theorem 2:1 gives the solution

(2:42)
$$\begin{aligned}
x &= \alpha^2 + 4\alpha\beta + 2\beta^2 - 4\alpha\gamma - 2\beta\gamma + 2\beta\delta - 4\gamma\delta, \\
y &= -4\alpha^2 - 9\alpha\beta - 4\beta^2 + 2\alpha\gamma - 2\beta\delta + 4\gamma\delta, \\
z &= -2\alpha^2 - 3\alpha\beta - \beta^2 - 2\alpha\gamma - \beta\gamma - 2\gamma^2 - \beta\delta + 2\gamma\delta, \\
w &= -2\alpha^2 - 3\alpha\beta - \beta^2 + \alpha\gamma + \beta\gamma - 3\alpha\delta - 3\beta\delta
\end{aligned}$$

Now (2:42) gives a solution equivalent to any existing one which is not a solution of

(2:43) $$3x + 2y + 2z = 0.$$

But (2:37), (2:40), (2:41), and (2:43) have no solutions in common. Hence given any solution of (2:37) we may find a solution equivalent to it from either (2:38), (2:39), or (2:42).

CHAPTER III

CUBIC EQUATIONS

Several methods have been given for solving the equation[7/]

7/ Leonard E. Dickson, *op. cit.*, pp. 554 - 561. Robert D. Carmichael, *op. cit.*, pp. 62 - 66.

$x^3 + y^3 = u^3 + v^3$. Here a more general equation is solved, of which this is a special case. This is given in the following theorem.

Theorem 3:1. The equation[8/]

8/ W. V. Parker and A. A. Aucoin, "Solution of a Cubic Diophantine Equation," *Tohoku Mathematical Journal*, 45, 324-328, part 2, March, 1939.

$$(3:1) \qquad (ax + by)(mx^2 + nxy + py^2) = (cu + dv)(mu^2 + nuv + pv^2)$$

has solutions and every solution which is not also a solution of

$$(3:2) \qquad \frac{(mx^2 + nxy + py^2)^3}{cu + dv}\Big\{[(bc - ad)mx + (bcn - acp - bdm)y]u + [(bdm - adn + acp)x + (bc - ad)py]v\Big\} = 0$$

is equivalent to one of the solutions given by

$$(3:3) \qquad \begin{aligned} x &= \gamma[b\gamma^3 - (\alpha^2 + n\alpha\beta + mp\beta^2)(d\alpha + dn\beta - cp\beta)], \\ y &= \gamma[(\alpha^2 + n\alpha\beta + mp\beta^2)(c\alpha + dm\beta) - a\gamma^3], \\ u &= (b\alpha + ap\beta)\gamma^3 - d(\alpha^2 + n\alpha\beta + mp\beta^2)^2, \\ v &= (mb\beta - a\alpha - an\beta)\gamma^3 + c(\alpha^2 + n\alpha\beta + mp\beta^2)^2, \end{aligned}$$

where α, β, γ are arbitrary integers.

Proof. Write (3:1) in the form

$$(3:4) \qquad \frac{ax + by}{cu + dv} = \frac{mu^2 + nuv + pv^2}{mx^2 + nxy + py^2}.$$

Since $\gamma^2(mu^2 + nuv + pv^2) = (mx^2 + nxy + py^2)(\alpha^2 + n\alpha\beta + mp\beta^2)$

if

$$\gamma u = \alpha x - p\beta y, \qquad \gamma v = m\beta x + (\alpha + n\beta)y, \tag{3:5}$$

$$\frac{ax + by}{cu + dv} = \frac{mu^2 + nuv + pv^2}{mx^2 + nxy + py^2} = \frac{\alpha^2 + n\alpha\beta + mp\beta^2}{\gamma^2} \tag{3:6}$$

may be written in place of (3:4). If $cu + dv = \delta$, from (3:5) and (3:6) it follows that

$$(3{:}7)\qquad \begin{aligned} (c\alpha + dm\beta)x + [d(\alpha + n\beta) - cp\beta]y &= \gamma\delta, \\ a\gamma^2 x + b\gamma^2 y &= (\alpha^2 + n\alpha\beta + mp\beta^2)\delta. \end{aligned}$$

Let

$$\delta = \gamma \begin{vmatrix} c\alpha + dm\beta & d(\alpha + n\beta) - cp\beta \\ a\gamma^2 & b\gamma^2 \end{vmatrix}$$

and solve (3:7) for x and y. With these values, u and v may be determined from (3:5), and it is seen that (3:3) is a solution of (3:1).

If (x_1, y_1) is any solution of $ax + by = 0$ or $mx^2 + nxy + py^2 = 0$ and (u_1, v_1) is any solution of $cu + dv = 0$ or $mu^2 + nuv + pv^2 = 0$, then (x_1, y_1, u_1, v_1) is a trivial solution of (3:1). Such solutions will not be considered.

Suppose that $x = \lambda$, $y = \mu$, $u = \rho$, $v = \sigma$ is any non-trivial solution of (3:1). Choose $\alpha = m\lambda\rho + n\mu\rho + p\mu\sigma$, $\beta = \lambda\sigma - \mu\rho$, $\gamma = m\lambda^2 + n\lambda\mu + p\mu^2$, then

$$\begin{aligned} \alpha^2 + n\alpha\beta + mp\beta^2 &= (m\lambda^2 + n\lambda\mu + p\mu^2)(m\rho^2 + n\rho\sigma + p\sigma^2) \\ &= (m\lambda^2 + n\lambda\mu + p\mu^2)^2 \left[\frac{a\lambda + b\mu}{c\rho + d\sigma}\right], \end{aligned}$$

so that when these values are substituted in (3:3) it follows that $x = \lambda K$, $y = \mu K$, $u = \rho K$, $v = \sigma K$, where K is the left hand member of (3:2) with x, y, u, v replaced by $\lambda, \mu, \rho, \sigma$. Hence every solution of (3:1) which does not satisfy (3:2) is equivalent to a solution given by (3:3).

The following observation is made concerning the solution. If a non-trivial solution does satisfy (3:2), then[9/]

9/ Since the solution is non-trivial, $mx^2 + nxy + py^2 \neq 0$.

$$(3{:}8)\qquad \begin{aligned} u &= S[(adn - acp - bdm)x - (bc - ad)py], \\ v &= S[(bc - ad)mx + (bcn - acp - bdm)y]. \end{aligned}$$

If the values from (3:8) are substituted in (3:1), then

$$(3{:}9)\qquad (ax + by)(mx^2 + nxy + py^2) = MS^3(ax + by)(mx^2 + nxy + py^2)$$

where

$$M = (c^2p + d^2m - cdn)\begin{vmatrix} adn - acp - bdm & (ad - bc)p \\ (bc - ad)m & bcn - acp - bdm \end{vmatrix}.$$

Any values (x, y) will satisfy (3:9) if $MS^3 = 1$ and no value if $MS^3 \neq 1$. Therefore, in addition to the values given by (3:3), (3:1) is satisfied by all integral solutions of

$$(3{:}10)\qquad \begin{aligned} \sqrt[3]{M}\, u &= (adm - acp - bdm)x - (bc - ad)py, \\ \sqrt[3]{M}\, v &= (bc - ad)mx + (bcn - acp - bdm)y, \end{aligned}$$

if (3:10) has any integral solutions.

For the equation $x^3 + y^3 = u^3 + v^3$, $a = b = c = d = m = p = -n = 1$ and (3:3) becomes

$$\begin{aligned} x &= \gamma[\gamma^3 - (\alpha - 2\beta)(\alpha^2 - \alpha\beta + \beta^2)], \\ y &= \gamma[(\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) - \gamma^3], \\ u &= (\alpha + \beta)\gamma^3 - (\alpha^2 - \alpha\beta + \beta^2)^2, \\ v &= (\alpha^2 - \alpha\beta + \beta^2)^2 - (\alpha - 2\beta)\gamma^3. \end{aligned}$$

By the use of a multiplicative domain, Carmichael[10/]

10/ Robert D. Carmichael, *op. cit.*, p. 62, formulas (4) and (5).

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obtains a solution of the equation

$$(3{:}11)\qquad x^3 + y^3 + z^3 - 3xyz = u^3 + v^3 + w^3 - 3uvw.$$

He says, "This solution, so readily obtained, unfortunately lacks generality." By a second method he gets expressions for x, y, u, v in terms of arbitrary parameters a, b, z, w. These are not shown to give the general solution of (3:11).

A more general problem will now be solved which gives the general solution of (3:11) as a special case. Let $f(x) = f(x_1, \ldots, x_n)$ be a homogeneous polynomial of degree three with integral coefficients. Suppose $f(x)$ is such that for a set of integers $a_1, a_2, \ldots, a_n$, not all zero,

$$\frac{\partial f}{\partial x_j} = 0, \text{ for } x_i = a_i \,, \quad (i, j = 1, \ldots, n).$$

If $x_i = a_i s + \alpha_i t$, then

$$f(x) = s^3 f(a) + s^2 t \sum_1^n \alpha_j \frac{\partial f}{\partial a_j} + st^2 \sum_1^n a_j \frac{\partial f}{\partial \alpha_j} + t^3 f(\alpha),$$

and since $\frac{\partial f}{\partial a_j} = 0$, $f(a) = \frac{1}{3} \sum_1^n a_j \frac{\partial f}{\partial a_j} = 0$, and hence

$$f(x) = st^2 \sum_{j=1}^n a_j \frac{\partial f}{\partial \alpha_j} + t^3 f(\alpha).$$

Theorem 3:2. The equation[11]

11/ W. V. Parker and A. A. Aucoin, "On Cubic Diophantine Equations," National Mathematics Magazine, 13:115 - 117, December, 1938.

(3:12) $$f(x) = g(y),$$

where $g(y) = g(y_1, \ldots, y_m)$ is a homogeneous polynomial of degree three with integral coefficients, has solutions and every solution which is not also a solution of

(3:13) $$\sum_1^m a_j \frac{\partial f}{\partial x_j} = 0,$$

is equivalent to one of the solutions given by

(3:14) $$x_i = a_i s + \alpha_i t, \qquad y_j = \beta_j t,$$

where s and t are given by

(3:15) $$s = g(\beta) - f(\alpha), \quad t = \sum_j^n a_j \frac{\partial f}{\partial \alpha_j},$$

and the α's and β's are arbitrary integers.

Proof. If x and y have the values given by (3:14), then (3:12) becomes $st^2 \sum_j^n a_j \frac{\partial f}{\partial \alpha_j} = t^2[g(\beta) - f(\alpha)]$ which is identically satisfied in the α's and β's if s and t have the values given by (3:15).

Let $x_i = \lambda_i$, $y_j = \mu_j$ be any given solution of (3:12). If the choice $\alpha_i = \lambda_i$, $\beta_j = \mu_j$ is made in (3:15), then since $s = 0$, the solution becomes $x_i = \lambda_i t$, $y_j = \mu_j t$ which is equivalent to the given solution provided $t \neq 0$; that is, provided $x_i = \lambda_i$, $y_j = \mu_j$ does not satisfy (3:13).

Consider now equation (3:11). Here $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$, for $x = y = z = 1$. Hence the solution is

(3:16)
$$\begin{aligned}
x &= (\lambda^3 + \mu^3 + \nu^3 - 3\lambda\mu\nu) + (\alpha - \beta)^3 + (\alpha - \gamma)^3, \\
y &= (\lambda^3 + \mu^3 + \nu^3 - 3\lambda\mu\nu) + (\beta - \alpha)^3 + (\beta - \gamma)^3, \\
z &= (\lambda^3 + \mu^3 + \nu^3 - 3\lambda\mu\nu) \quad (\gamma - \alpha)^3 + (\gamma - \beta)^3, \\
u &= 3\lambda(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma), \\
v &= 3\mu(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma), \\
w &= 3\nu(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma).
\end{aligned}$$

Now (3:16) gives a solution equivalent to any solution of (3:11) if it does not satisfy

(3:17) $$x^2 + y^2 + z^2 - xy - xz - yz = 0.$$

It is now shown that (3:17) has no solution other than $x = y = z$. Suppose $x = \alpha$, $y = \beta$, $z = \gamma$ is any solution of (3:17). Then $x = \alpha - 1$, $y = \beta - 1$, $z = \gamma - 1$ is also a solution. Now there is no loss of generality in supposing that one of the numbers, say α, is positive. By continuing to substract one from the values of x, y, z a solution is obtained in which $x = 0$ and y and z must be a solution of $y^2 - yz + z^2 = 0$. But this equation has no solutions other than $y = z = 0$. Hence $x = y = z = \alpha$ is the only solution of (3:17). In has been shown, incidentally, that all the solutions of the equation $x^3 + y^3 + z^3 - 3xyz = 0$ except the trivial one $x = y = z$ are solutions of $x + y + z = 0$. These[12] may be obtained from (3:16)

12/ Leonard E. Dickson, *op. cit.*, p. 590.

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by setting $\lambda = \mu = \nu = 0$. Or it is seen by inspection that $x + y + z = 0$ is satisfied by $x = \alpha$, $y = \beta$, $z = -(\alpha + \beta)$.

CHAPTER IV

QUARTIC EQUATIONS

One function f which satisfies the conditions of Theorem 3:2 is the product of three linear factors in three unknowns. The next consideration will be an equation in which the left hand member is the product of four linear factors in three unknowns.

Theorem 4:1. The equation

$$\prod_{i}^{4} \sum_{j}^{3} a_{ij}\, x_j = g(y), \tag{4:1}$$

where $g(y) = g(y_1, \ldots, y_p)$ is a homogeneous polynomial of degree n, $n \not\equiv 0 \pmod 4$, has solutions and every solution for which the members of (4:1) do not vanish is equivalent to either a solution given by

$$\begin{aligned} x_j &= A^{n-1} s^m t^m \left[t(\alpha_1 A_{1j} + \alpha_2 A_{2j}) + s\, \alpha_3 A_{3j} \right], \\ y_k &= A^4\, s^2 t^2 \beta_k , \end{aligned} \tag{4:2}$$

or a solution given by

$$\begin{aligned} x_j &= A^{n-1} s^m t^m \left[t(\alpha_1 A_{1j} + \alpha_2 A_{2j}) + s\, \alpha_3 A_{3j} \right], \\ y_k &= A^4\, st\, \beta_k , \end{aligned} \tag{4:3}$$

according as $n = 2m+1$ or $n = 4m+2$, where $A\ (\neq 0)$ is the determinant of the forms $\sum_{j}^{3} a_{ij} x_j$ $(i = 1,2,3)$ and A_{ij} is the cofactor of a_{ij} in A, s and t are given by

$$\begin{aligned} s &= \alpha_1 \alpha_2 \alpha_3 \sum_{j}^{3} a_{4j} (\alpha_1 A_{1j} + \alpha_2 A_{2j}), \\ t &= Ag(\beta) - \alpha_1 \alpha_2 \alpha_3^2 \sum_{j}^{3} a_{4j} A_{3j} , \end{aligned} \tag{4:4}$$

the α's and β's being arbitrary integers.

Proof. Let

$$\sum_{j}^{3} a_{3j}\, x_j = A^n \alpha_3\, s^{m+1} t^m , \qquad \sum_{j}^{3} a_{ij}\, x_j = A^n \alpha_i\, s^m t^{m+1} \quad (i = 1,2). \tag{4:5}$$

The solution of this system of equations is

(4:6) $$x_j = A^{n-1} s^m t^m \left[t(\alpha_1 A_{1j} + \alpha_2 A_{2j}) + s\,\alpha_3 A_{3j} \right]$$

and hence

(4:7) $$\sum_j^3 a_{4j} x_j = A^{n-1} s^m t^m \left[t \sum_j^3 a_{4j} (\alpha_1 A_{1j} + \alpha_2 A_{2j}) + s\,\alpha_3 \sum_j^3 a_{4j} A_{3j} \right].$$

The theorem now divides itself into two cases. First let $n = 2m + 1$. If

(4:8) $$y_k = A^4 s^2 t^2 \beta_k,$$

then from (4:5), (4:7), and (4:8), (4:1) becomes

(4:9) $$A^{4n-1} s^{4m+1} t^{4m+2} \alpha_1\alpha_2\alpha_3 \left[t \sum_j^3 a_{4j} (\alpha_1 A_{1j} + \alpha_2 A_{2j}) + s\,\alpha_3 \sum_j^3 a_{4j} A_{3j} \right] = A^{4n} s^{4m+2} t^{4m+2} g(\beta),$$

which is identically satisfied in the α's and β's if s and t are given by (4:4).

Second, let $n = 4m + 2$. If

(4:10) $$y_k = A^4 s t \beta_k,$$

then (4:1) becomes (4:9) and hence as before s and t are given by (4:4).

Suppose now that $x_j = \lambda_j$, $y_k = \mu_k$ is any solution of (4:1). Choose $\alpha_i = \sum_k^3 a_{ik} \lambda_k$, $\beta_k = \mu_k$, and then

$$s = \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \sum_k^3 \sum_l^3 (a_{4l} a_{1k} A_{1l} + a_{4l} a_{2k} A_{2l}) \lambda_k$$

$$\begin{aligned} t &= Ag(\mu) - \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \sum_k^3 a_{3k} \lambda_k \sum_l^3 a_{4l} A_{3l} \\ &= \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \left\{ A \sum_k^3 a_{4k} \lambda_k - \sum_k^3 a_{3k} \lambda_k \sum_l^3 a_{4l} A_{3l} \right\} \\ &= \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \left\{ \sum_k^3 (A a_{4k} - a_{3k} \sum_l^3 a_{4l} A_{3l}) \lambda_k \right\}. \end{aligned}$$

and hence

$$\begin{aligned} s - t &= \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \left\{ \sum_k^3 \sum_l^3 (a_{4l} a_{1k} A_{1l} + a_{4l} a_{2k} A_{2l} + a_{3k} a_{4l} A_{3l}) - A \sum_k^3 a_{4k} \right\} \lambda_k \\ &= \left[\prod_i^3 \sum_j^3 a_{ij} \lambda_j \right] \left[A \sum_k^3 a_{4k} - A \sum_k^3 a_{4k} \right] \lambda_k \\ &= 0, \end{aligned}$$

and $s = t$. Therefore

$$x_j = t \sum_i^3 \alpha_i A_{ij}$$

$$= t \sum_{i}^{2} \sum_{k}^{2} a_{ik} A_{ij} \lambda_k$$

$$= tA \lambda_k .$$

In the first case $y_k = A^4 t^4 \beta_k$ and in the second case $y_k = A^4 t^2 \beta_k$ and so in either case a solution is obtained that is equivalent to any solution.

It is evident that this could be extended to 2n factors in 2n - 1 variables with the proper restrictions on g(y).

A special case of (4:1) is

$$(x + y + z)(-x + y + z)(x - y + z)(x + y - z) = 4A^2,$$

which is the formula for expressing the area A of a triangle in terms of its sides x, y, z. It is often reduced by a linear transformation[13] to

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13/ Robert D. Carnichael, *op. cit.*, p. 9.

$$uvw(u + v + w) = t^2$$

and then solved.

CHAPTER V

EQUATIONS OF DEGREE N

The theorems of Chapter II as well as Theorem 3:2 will now be generalized.

Suppose $f(x) = f(x_1, \ldots, x_p)$ is a homogeneous polynomial of degree n, with integral coefficients, and such that for $x_i = a_i$, a_i integral and not all zero, all its partial derivatives of all orders less than n - 1 vanish. If $x_i = a_i s + \alpha_i t$, then by Taylor's theorem [14/]

[14/] By Euler's theorem $f(a) = \frac{1}{n}\sum_{i}^{p} a_i \frac{\partial f}{\partial a_i} = 0$. It is also assumed that f is such that for this choice of values, $x_i = a_i$, $\sum_j a_j \frac{\partial f}{\partial x_j} \neq 0$.

$$f(x) = st^{n-1}\sum_j^p a_j \frac{\partial f}{\partial \alpha_j} + t^n f(\alpha).$$

Theorem 5:1. The equation [15/]

[15/] This theorem is contained in a paper by A. A. Aucoin, accepted for publication by the Bulletin of the American Mathematical Society. The theorem generalizes Theorems 2;1 and 3:2.

$$f(x) = g(y), \tag{5:1}$$

where $g(y) = g(y_1, \ldots, y_q)$ is a homogeneous polynomial of degree m, has solutions, and every solution which is not also a solution of

$$\sum_j^p a_j \frac{\partial f}{\partial x_j} = 0, \tag{5:2}$$

is equivalent to one of the solutions given by

$$\begin{aligned} x_i &= a_i s t^{\lambda-1} + \alpha_i t^\lambda, \\ y_k &= \beta_k t^\mu, \end{aligned} \tag{5:3}$$

where λ, μ are relatively prime positive integers such that

$\lambda n = \mu m$, α_i, β_k are arbitrary integers, and

$$s = g(\beta) - f(\alpha), \tag{5:4}$$
$$t = \sum_j^r a_j \frac{\partial f}{\partial \alpha_j}.$$

Proof. If in (5:1), x_i, y_k have the values given by (5:3), then

$$st^{n\lambda-1} \sum_j^r a_j \frac{\partial f}{\partial \alpha_j} + t^{n\lambda} f(\alpha) = t^{m\mu} g(\beta),$$

which is satisfied identically in the α's and β's when s and t are given by (5:4). Hence (5:3) affords a solution of (5:1).

Suppose $x_i = \rho_i$, $y_k = \sigma_k$ is any solution of (5:1). If $\alpha_i = \rho_i$, $\beta_k = \sigma_k$, $s = 0$ and (5:3) becomes $x_i = \rho_i t^\lambda$, $y_k = \sigma_k t^\mu$, which is equivalent to the given solution provided $x_i = \rho_i$, $y_k = \sigma_k$ is not also a solution of (5:2). It will be noted that g may be identically zero.

Corollary. The equation

$$f(x) = \sum_j^r x_j g_j(y) + g(y),$$

where $g_j(y) = g_j(y_1, \ldots, y_q)$ and $g(y) = g(y_1, \ldots, y_q)$ are homogeneous polynomials with integral coefficients of degrees $n - 1$ and n respectively, has solutions, and every solution which is not also a solution of

$$\sum_j^r a_j \left[\frac{\partial f}{\partial x_j} - g_j(y)\right] = 0$$

is equivalent to one of the solutions given by $x_i = a_i s + \alpha_i t$, $y_k = \beta_k t$, where $s = g(\beta) - f(\alpha) + \sum_j^r a_j g_j(\beta)$, $t = \sum_j^r a_j \left[\frac{\partial f}{\partial \alpha_j} - g_j(\beta)\right]$.

Theorem 5:2. The equation

$$f(x) \sum_j^r d_j x_j = R(y), \tag{5:5}$$

where f satisfies the hypothesis of Theorem 5:1 and $R(y) = R(y_1, \ldots, y_q)$ is a homogeneous polynomial with integral coefficients, of degree $n - 1$, has solutions and every solution which is not also a solution of

(5:6) $$f(x)\sum_{j}^{r} a_j \frac{\partial f}{\partial x_j}\left[\sum_{j}^{r} a_j \frac{\partial f}{\partial x_j}\sum_{j}^{r} d_j x_j - f(x)\sum_{j}^{r} d_j a_j\right] = 0$$

is equivalent to one of the solutions given by

(5:7) $$x_i = a_i s + \alpha_i t,$$
$$y_k = \beta_k t,$$

where

$$s = A^{n-1}\left[\lambda(AD - BC)\right]^{n-2}\left[D\lambda^2 - BR(\mu)\right],$$

(5:8) $$t = A^{n-1}\left[\lambda(AD - BC)\right]^{n-2}\left[AR(\mu) - C\lambda^2\right],$$

$$\beta_k = A^{n}\left[\lambda(AD - BC)\right]^{2}\mu_k,$$

where $A = \sum_{j}^{r} a_j \frac{\partial f}{\partial \alpha_j}$, $B = f(\alpha)$, $C = \sum_{j}^{r} d_j a_j$, $D = \sum_{j}^{r} d_j \alpha_j$, α_j, λ, μ_k being arbitrary integers.

Proof. Let x_i, y_k have the values given by (5:7). Then (5:5) becomes

$$(Ast^{n-1} + Bt^{n})(Cs + Dt) = t^{n-1}R(\beta),$$

which becomes, after dividing out the factor[16] t^{n-1},

16/ It will be shown later that $t \neq 0$.

(5:9) $$(As + Bt)(Cs + Dt) = R(\beta).$$

By Theorem 5:1 the solution of (5:9) is given by[17] (5:8).

17/ It will be shown later that (5:9) satisfies the hypothesis of Theorem 5:1.

Suppose $x_i = \rho_i$, $y_k = \sigma_k$ is any given solution of (5:5). Then if $\alpha_i = \rho_i$, $\mu_k = \sigma_k$, $\lambda = f(\rho)$, $s = 0$ and the solution becomes $x_i = \rho_i K^{n-1}$, $y_k = \sigma_k K^{n+1}$ where $K = A\lambda(AD - BC)$, which is proportional to the given solution provided $K \neq 0$; that is, provided $x_i = \rho_i$, $y_k = \sigma_k$ is not a solution of (5:6). It will be noted that if $K \neq 0$, then $t \neq 0$.

One function of interest which satisfies the hypothesis of Theorem 5:1 is the function $D(x) = |a_{ij} x_{ij}|$, a determinant of order n with a_{ij} integral such that not all the a's in any row or column are zero. If there is any element, say x_{kl}, which is distinct from all others, we may select $x_{kl} = 1$ and $x_{ij} = 0$ otherwise. For this choice all partial derivatives of all orders less than $n - 1$ vanish. It is interesting to note that in the solution, the form of the expression for all the x's is the same except for x_{kl}. This is illustrated by

$$D = \begin{vmatrix} x & 3y & z & 0 \\ u & 7v & 3w & p \\ q & u & 3x & z \\ 0 & 2w & 5z & 0 \end{vmatrix} = r^n,$$

the solution of which is $x = \alpha t^n$, $y = \beta t^n$, $z = \gamma t^n$, $u = \lambda t^n$, $v = st^{n-1} + \mu t^n$, $w = \nu t^n$, $p = \rho t^n$, $q = \sigma t^n$, $r = \pi t^n$, where $s = \pi^n - D'$, $t = \Delta$, D' being D with x, y, z, u, v, w, p, q, replaced by $\alpha, \beta, \gamma, \lambda, \mu, \nu, \rho, \sigma$ and

$$\Delta = \begin{vmatrix} \alpha & 0 & \gamma & 0 \\ \lambda & 7 & 3\nu & \rho \\ \sigma & 0 & 3\alpha & \gamma \\ 0 & 0 & 5\gamma & 0 \end{vmatrix}.$$

It is not necessary, however, that there be one element which is distinct from all others in order that this method apply to certain problems. For example, D may be the circulant, as it was in Theorem 3:2.

Another function which also satisfies the hypothesis of Theorem 5:1 is the function $\prod_i^n \sum_j^n a_{ij} x_j$, where a_{ij} are integers and the determinant $A = |a_{ij}| \neq 0$. For x_j may be chosen integral, so that n - 1 of the above factors vanish and hence for this choice of x_j, all partial derivatives of all orders less than n - 1 vanish. Hence the equation

$$\prod_i^n \sum_j^n a_{ij} x_j = g(y),$$

where g(y) is a homogeneous polynomial with integral coefficients, of degree m, may be solved by means of Theorem 5:1. This equation may, however, be solved by an entirely different method. The method will be illustrated by solving a more general problem.

Let $P(x) = \sum_k^p \prod_i^n \sum_j^n a_{ij,k} x_{jk}$, where $a_{ij,k}$ are integers and the determinant A_k, for each k, of the forms $\sum_j^n a_{ij,k} x_{jk}$ is not zero. Without loss of generality, the notation may be so chosen that $A_{nn,k} \neq 0$, where $A_{ij,k}$ is the cofactor of $a_{ij,k}$ in A_k. Let $A^{(j)}_{nn,k}$ be the determinant obtained from $A_{nn,k}$ by replacing the jth column by $\alpha_{1,k}, \alpha_{2,k}, \ldots, \alpha_{n-1,k}$, R the least common multiple of $A_{nn,k}$ for each k, and $R_k = R/A_{nn,k}$.

Theorem 5:3. The equation

$$P(x) = g(y), \tag{5:10}$$

where g(y) is a homogeneous polynomial with integral coefficients, of degree m, has solutions and every solution which is not also a solution of

$$\sum_k^p \prod_i^{n-1} \sum_j^n a_{ij,k} x_{jk} R_k x_{nk} A_k = 0, \tag{5:11}$$

is equivalent to one of the solutions given by

$$\begin{aligned} x_{jk} &= t^\lambda R^{\lambda-1} R_k A^{(j)}_{nn,k} + st^{\lambda-1} R^{\lambda-1} R_k \beta_k A_{nj,k}, \quad \begin{pmatrix} j = 1, \ldots, n-1 \\ k = 1, \ldots, p \end{pmatrix} \\ x_{nk} &= st^{\lambda-1} R^\lambda \beta_k, \quad (k = 1, \ldots, p) \\ y_j &= \gamma_j t^\mu R^\mu, \end{aligned} \tag{5:12}$$

where

$$\text{(5:13)}\qquad \begin{aligned} s &= Rg(\gamma) - \sum_k^p \Big[\prod_i^{n-1} \alpha_{i,k} R_k\Big] \sum_j^{n-1} a_{nj,k} A^{(j)}_{nn,k}, \\ t &= \sum_k^p \prod_i^{n-1} \alpha_{i,k} R_k \beta_k A_k, \end{aligned}$$

and the α's, β's, and γ's are arbitrary integers, λ, μ, are relatively prime positive integers such that $\lambda n = \mu m$.

Proof. Let

$$\text{(5:14)}\qquad \sum_j^n a_{ij,k} x_{jk} = t^\lambda R^\lambda \alpha_{i,k} \qquad \begin{pmatrix} i = 1,..,n-1 \\ k = 1,..,p \end{pmatrix}$$

and set

$$\text{(5:15)}\qquad x_{nk} = s t^{\lambda-1} R^\lambda \beta_k. \qquad (k = 1,..,p)$$

Now write (5:14) as

$$\text{(5:16)}\qquad \sum_j^{n-1} a_{ij,k} x_{jk} = t^\lambda R^\lambda \alpha_{i,k} - s t^{\lambda-1} R^\lambda \beta_k a_{in,k}. \qquad \begin{pmatrix} i = 1,..,n-1 \\ k = 1,..,p \end{pmatrix}$$

Solving (5:16) gives

$$\text{(5:17)}\qquad x_{jk} = t^\lambda R^{\lambda-1} R_k A^{(j)}_{nn,k} + s t^{\lambda-1} R^{\lambda-1} R_k \beta_k A_{nj,k}$$

and hence

$$\text{(5:18)}\qquad \sum_j^n a_{nj,k} x_{jk} = t^{\lambda-1} R^{\lambda-1} R_k \Big[t \sum_j^{n-1} a_{nj,k} A^{(j)}_{nn,k} + s \beta_k A_k\Big].$$

If $y_j = t^\mu R^\mu \gamma_j$, then by use of (5:14) and (5:18), (5:10) becomes

$$t^{n\lambda-1} R^{n\lambda-1} \sum_k^p \Big[\prod_i^{n-1} \alpha_{ik} R_k\Big]\Big[t \sum_j^{n-1} a_{nj,k} A^{(j)}_{nn,k} + s \beta_k A_k\Big] = t^{m\mu} R^{m\mu} g(\gamma)$$

which is identically satisfied in the α's, β's, and γ's if s and t are given by (5:13). Hence (5:12) is a solution of (5:10).

Suppose now that $x_{jk} = \rho_{jk}$, $y_j = \sigma_j$ is a given solution of (5:10). Choose $\alpha_{ik} = \sum_j^n a_{ij,k} \rho_{jk}$, $\beta_k = \rho_{nk}$, $\gamma_j = \sigma_j$ then [18]

18/ $A^{(j)}_{nn,k}$ becomes $\rho_{jk} A_{nn,k} - \rho_{nk} A_{nj,k}$ when $\alpha_{i,k}$ is replaced by $\sum_j^n a_{ij,k} \rho_{jk}$.

$$t = \sum_k^p \Big[\prod_i^{n-1} \sum_j^n a_{ij,k} \rho_{jk}\Big] \rho_{nk} A_k R_k.$$

$$\begin{aligned} s &= Rg(\gamma) - \sum_k^p \Big[\prod_i^{n-1} \alpha_{i,k} R_k\Big] \sum_j^{n-1} a_{nj,k} A^{(j)}_{nn,k} \\ &= \sum_k \Big[R \prod_i^n \sum_j^n a_{ij,k} \rho_{jk} - \prod_i^{n-1} \Big[\sum_j^n a_{ij,k} \rho_{jk}\Big] R_k \sum_j^{n-1} a_{nj,k} (\rho_{jk} A_{nn,k} - \rho_{nk} A_{nj,k})\Big] \end{aligned}$$

$$
\begin{aligned}
s &= \sum_{k}^{p}\Big[\Big\{\prod_{i}^{n-1}\sum_{j}^{n} a_{ij,k}\rho_{jk}\Big\}\Big\{R\sum_{l}^{n} a_{nl,k}\rho_{lk} - R_k\sum_{l}^{n-1} a_{nl,k}(\rho_{lk}A_{nn,k} - \rho_{nk}A_{nl,k})\Big\}\Big] \\
&= \sum_{k}^{p}\Big[\Big\{\prod_{i}^{n-1}\sum_{j}^{n} a_{ij,k}\rho_{jk}\Big\}\Big\{R a_{nn,k}\rho_{nk} + R_k\sum_{l}^{n-1} a_{nl,k}A_{nl,k}\rho_{nk}\Big\}\Big] \\
&= \sum_{k}^{p}\Big[\Big\{\prod_{i}^{n-1}\sum_{j}^{n} a_{ij,k}\rho_{jk}\Big\} A_k R_k \rho_{nk}\Big] \\
&= t.
\end{aligned}
$$

Hence

$$
\begin{aligned}
x_{jk} &= t^{\lambda}R^{\lambda-1}R_k[\rho_{jk}A_{nn,k} - \rho_{nk}A_{nj,k} + \rho_{nk}A_{nj,k}] \\
&= \rho_{jk}t^{\lambda}R^{\lambda}, \\
x_{nk} &= \rho_{nk}t^{\lambda}R^{\lambda}, \\
y_j &= \sigma_j t^{\mu}R^{\mu},
\end{aligned}
$$

which is equivalent to the given solution provided $x_{jk} = \rho_{jk}$, $y_j = \sigma_j$ is not also a solution of (5:11).

Let $f(x,y) = f(x_1, \ldots, x_a; y_1, \ldots, y_a)$, $g(x,y) = g(x_1, \ldots, x_a; y_1, \ldots, y_a)$ be polynomials with integral coefficient, homogeneous in both x and y. Suppose that $f(x,y)$ is of degree m in x, q in y, and $g(x,y)$ is of degree n in x and p in y, where m, n, p, q are positive integers and $m - n > 0$ is relatively prime to $p - q > 0$. Since $m - n$ and $p - q$ are relatively prime, there exist non-negative integers u, v, $u \leq p - q$, $v \leq m - n$, such that $(m - n)u = (p - q)v + 1$ and $(m - n)(p - q - u) = (p - q)(m - n - v) - 1$.

Theorem 5:4. The equation[19]

19/ For a special case of this theorem see, A. A. Aucoin and W. V. Parker, "Diophantine Equations whose Members are homogenrous," Bulletin of the American Mathematical Society, 45: 330 - 333, April, 1939.

$$(5{:}20) \qquad f(x,y) = g(x,y)$$

has solutions, and every solution for which the members do vanish, is equivalent to one of the solutions given by

$$
(5{:}21) \qquad \begin{aligned} x_i &= \alpha_i [g(\alpha,\beta)]^{u}\,[f(\alpha,\beta)]^{p-q-u}, \\ y_j &= \beta_j [g(\alpha,\beta)]^{v}\,[f(\alpha,\beta)]^{m-n-v}, \end{aligned}
$$

where α_i, β_j are arbitrary integers.

Proof. Let $x_i = \alpha_i s^{u} t^{p-q-u}$, $y_j = \beta_j s^{v} t^{m-n-v}$. Then (5:20) becomes $s^{mu+qv} t^{m(p-q-u)+q(m-n-v)} f(\alpha,\beta) = s^{nu+pv} t^{n(p-q-u)+p(m-n-v)} g(\alpha,\beta)$, which is satisfied identically in the α's and β's if $s = g(\alpha,\beta)$, $t = f(\alpha,\beta)$. Hence (5:21) is a solution of (5:20).

If $x_i = \lambda_i$, $y_j = \mu_j$ is a given solution of (5:20) then $f(\lambda,\mu) = g(\lambda,\mu)$. If $\alpha_i = \lambda_i$, $\beta_j = \mu_j$ then $x_i = \lambda_i [f(\lambda,\mu)]^{p-q}$, $y_j = \mu_j [f(\lambda,\mu)]^{m-n}$, which is equivalent to the given solution provided $f(\lambda,\mu) \neq 0$.

The proof of Theorem 5:4 is the same if $m - n < 0$ and $p - q < 0$. The method does not apply, however, in the cases $m - n > 0$, $p - q < 0$ and $m - n < 0$, $p - q > 0$.

Suppose now that for the functions f and g considered above, f is of degree m in x, $-p$ in y; g is of degree n in x, $-q$ in y, m, n, p, q, are positive integers and $m - n > 0$ is relatively prime to $p - q > 0$. As before, non-negative integers u, v exist such that $(m - n)u = (p - q)v + 1$ and $(m - n)(p - q - u) = (p - q)(m - n - v) - 1$.

Theorem 5:5. The equation

$$
(5{:}22) \qquad f(x,y) = g(x,y)
$$

has solutions and every solution for which the members of (5:22) do not vanish is equivalent to a solution given by

$$
(5{:}23) \qquad \begin{aligned} x_i &= \alpha_i [A(\beta)B(\beta)]^{p-q}\,[g(\alpha,\beta)]^{u}\,[f(\alpha,\beta)]^{p-q-u}, \\ y_j &= \beta_j [A(\beta)B(\beta)]^{m-n}\,[g(\alpha,\beta)]^{v}\,[f(\alpha,\beta)]^{m-n-v}, \end{aligned}
$$

where $A(\beta) = \prod_j^{a} \beta_j^{p}$, $B(\beta) = \prod_j^{a} \beta_j^{q}$ and the α's and β's are arbitrary integers.

Proof. If (5:22) is multiplied by $A(y)B(y)$ it becomes

$$(5:24) \qquad A(y)B(y)f(x,y) = A(y)B(y)g(x,y),$$

each member of which is a polynomial with integral coefficients. If $x_i = \alpha_i s^u t^{p-q-u}$, $y_j = \beta_j s^v t^{m-n-v}$, then on substituting in (5:24) it follows that

$$s^{apv+aqv+mu-pv}\, t^{ap(m-n-v)+aq(m-n-v)+m(p-q-u)-p(m-n-v)} A(\beta)B(\beta)f(\alpha,\beta)$$
$$= s^{apv+aqv+nu-qv}\, t^{ap(m-n-v)+aq(m-n-v)+n(p-q-u)-q(m-n-v)} A(\beta)B(\beta)g(\alpha,\beta),$$

which is satisfied identically in the α's and β's if $s = A(\beta)B(\beta)g(\alpha,\beta)$, $t = A(\beta)B(\beta)f(\alpha,\beta)$. Hence (5:23) is a solution of (5:22).

If $x_i = \lambda_i$, $y_j = \mu_j$ is a given solution of (5:22) then $f(\lambda,\mu) = g(\lambda,\mu)$. If $\alpha_i = \lambda_i$, $\beta_j = \mu_j$ the solution becomes $x_i = \lambda_i[A(\mu)B(\mu)f(\lambda,\mu)]^{p-q}$, $y_j = \mu_j[A(\mu)B(\mu)g(\lambda,\mu)]^{m-n}$, which is equivalent to the given solution provided $f(\lambda,\mu) \neq 0$. It is necessary that $\mu_j \neq 0$.

It will be noted that if in Theorem 5:5, $m - n > 0$ and $p - q < 0$, then u, v may be chosen as positive integers such that $(m - n)u = (q - p)v + 1$ and $(m - n)(q - p - u) = (q - p)(m - n - v) - 1$ and the solution in this case is

$$x_i = \alpha_i[A(\beta)B(\beta)]^{q-p}\,[g(\alpha,\beta)]^u\,[f(\alpha,\beta)]^{q-p-u},$$
$$y_j = \beta_j[A(\beta)B(\beta)]^{m-n}\,[g(\alpha,\beta)]^v\,[f(\alpha,\beta)]^{m-n-v}.$$

If in Theorem 5:4 f is of degree $-m$ in x, $-q$ in y and g is of degree $-n$ in x and $-p$ in y the solution is

$$x_i = \alpha_i[A(\beta)B(\beta)C(\alpha)D(\alpha)]^{p-q}\,[f(\alpha,\beta)]^u\,[g(\alpha,\beta)]^{p-q-u},$$
$$y_j = \beta_j[A(\beta)B(\beta)C(\alpha)D(\alpha)]^{m-n}\,[f(\alpha,\beta)]^v\,[g(\alpha,\beta)]^{m-n-v},$$

where $C(\alpha) = \prod_j^a \alpha_j^m$, $D(\alpha) = \prod_j^a \alpha_j^n$.

The above methods may be extended to solve certain types of non-homogeneous equations, as is illustrated in the next theorem.

Theorem 5:6. The equation

$$f(x,y) = g(x,y), \tag{5:25}$$

where $f(x,y) = \sum_i^m a_i \prod_j^n x_j^{\alpha_{ij}} y_j^{\beta_{ij}}$, $g(x,y) = \sum_i b_i \prod_j x_j^{\gamma_{ij}} y_j^{\delta_{ij}}$, a_i, b_i are integers, α_{ij}, β_{ij}, γ_{ij}, δ_{ij} being positive integers, has solutions if there exists positive integers u_i, v_i, w_i, z_i, M, N such that

$$\begin{aligned} \sum_j^n(\alpha_{ij}u_j + \beta_{ij}w_j) &= M, \\ \sum_j^n(\gamma_{ij}u_j + \delta_{ij}w_j) &= M+1, \\ \sum_j^n(\gamma_{ij}v_j + \delta_{ij}z_j) &= N, \\ \sum_j^n(\alpha_{ij}v_j + \beta_{ij}z_j) &= N+1, \end{aligned} \tag{5:26}$$

and every solution for which the members of (5:25) do not vanish, is equivalent to one of the solutions given by

$$x_i = \alpha_i s^{u_i} t^{v_i}, \qquad y_j = \beta_j s^{w_j} t^{z_j}, \tag{5:27}$$

where

$$s = f(\alpha, \beta), \qquad t = g(\alpha, \beta), \tag{5:28}$$

and the α_i's and β_j's are arbitrary integers.

Proof. If x_i, y_j have the values given by (5:27) then by (5:26), (5:25) becomes $s^M t^{N+1} f(\alpha, \beta) = s^{M+1} t^N g(\alpha, \beta)$, which is satisfied identically in the α_i's and β_j's if s and t are given by (5:28).

If (5:26) is satisfied, there exist non-negative integers K, u_j', w_j' such that

$$\begin{aligned} \sum_j^n(\alpha_{ij}u_j' + \beta_{ij}w_j') &= K, \quad (i = 1,\ldots,m), \\ \sum_j^n(\gamma_{hj}u_j' + \delta_{hj}w_j') &= K, \quad (h = 1,\ldots,p). \end{aligned} \tag{5:29}$$

If $x_i = \lambda_i$, $y_j = \mu_j$ is any integral solution of (5:25) and there are no integers u_i', w_j' satisfying (5:29), $d > 1$, and λ_i', μ_j' such that $\lambda_i = \lambda_i' d^{u_i'}$, $\mu_j = \mu_j' d^{w_j'}$, then $x_i = \lambda_i$, $y_j = \mu_j$, is said to be a primitive solution of (5:25).

If $x_i = \lambda_i$, $y_j = \mu_j$ is a primitive solution of (5:25) then $x_i = \lambda_i d^{u_i'}$, $y_j = \mu_j d^{w_j'}$, <u>derived</u> from the primitive solution, where $d \neq 0$ is an integer and u_i', w_j' are non-negative integers satisfying (5:29), is also a solution. Two solutions are said to be equivalent if they may be derived from the same primitive solution.

With this definition we may show that for any solution of (5:25) for which the members do not vanish, an equivalent solution is given by (5:27).

If $x_i = \lambda_i$, $y_j = \mu_j$ is any given solution of (5:25) and the choice $\alpha_i = \lambda_i$, $\beta_j = \mu_j$ is made then

(5:30) $$x_i = \lambda_i [f(\lambda, \mu)]^{u_i + v_i}, \quad y_j = \mu_j [f(\lambda, \mu)]^{w_j + z_j}.$$

Since

$$\begin{aligned} &\sum_j^m [\alpha_{ij}(u_j + v_j) + \beta_{ij}(z_j + w_j)] \\ &= \sum_j^n [\gamma_{ij}(u_j + v_j) + \delta_{ij}(z_j + w_j)] \\ &= M + N + 1, \end{aligned}$$

the solution is equivalent to the given solution if $f(\lambda, \mu) \neq 0$.

Theorem 5:6 may be extended, as was Theorem 5:4, to cover equations in which the members are no longer polynomials. The method will be illustrated by a particular example. Consider

$$\frac{ax^3u^5}{v^7} + \frac{by^4v^4}{u^2} = \frac{cx^5}{w}$$

which may be written as

$$ax^3u^7w + by^4v^{11}w = cx^5u^2v^7.$$

The solution of this equation is $x = \alpha s^4 t^4$, $y = \beta s^3 t^3$, $u = \lambda s^{11} t^{11}$, $v = \mu s^7 t^7$, $w = \nu s t^3$ where $s = a \alpha^3 \lambda^7 \nu + b \beta^4 \mu^{11} \nu$, $t = c \alpha^5 \lambda^2 \mu^7$. It is necessary that u, v, and w be different from zero.

If $x = x'$, $y = y'$, $u = u'$, $v = v'$, $w = w'$ is a solution of the above equation and the choice $\alpha = x'$, $\beta = y'$, $\lambda = u'$, $\mu = v'$, $\nu = w'$ is made then $s = t$ and the solution becomes $x = x't^8$, $y = y't^6$, $u = u't^{22}$, $v = v't^{14}$, $w = w't^4$ which, by a natural extension of the definition of equivalence, is equivalent to the given solution.

Two final theorems will now be proved. Both of these theorems are included in the above theory, but because of their interest they will be solved in detail.

Theorem 5:7. The equation[20]

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[20] This theorem is to be published in the National Mathematics Magazine at a future date.

$$(5:31) \qquad \sum_{j}^{m} \frac{a_j}{\sum_i \frac{b_{ij}}{x_{ij}}} = \frac{c}{\sum_k \frac{c_k}{y_k}}$$

where all the letters represent integers[21] and x_{ij}, y_k are

---

[21] Values of the coefficients which will make some denominator vanish are excluded.

---

unknown, has integral solutions and every solution is equivalent to one of the solutions given by

$$(5:32) \qquad x_{ij} = \alpha_{ij} C(\beta) B(\alpha), \qquad y_k = \beta_k D(\beta) \sum_{j}^{m} \frac{a_j A_j(\alpha) B(\alpha)}{B_j(\alpha)},$$

where $A_j(\alpha) = \prod_i^n \alpha_{ij}$, $B_j(\alpha) = \sum_i^n \frac{b_{ij} A_j(\alpha)}{\alpha_{ij}}$, $B(\alpha) = \prod_j^m B_j(\alpha)$, $C(\beta) = \prod_k^p \beta_k$, $D(\beta) = \sum_k^p \frac{c_k C(\beta)}{\beta_k}$, and α_{ij}, β_k $(\neq 0)$ are arbitrary integers.

Proof. Equation (5:31) may be written in the form

$$D(y) \sum_{j}^{m} \frac{a_j A_j(x) B(x)}{B_j(x)} = cC(y)B(x). \tag{5:33}$$

If $x_{ij} = \alpha_{ij} t$, $y_k = \beta_k s$, (5:33) becomes

$$s^{r-1} t^{m(n-1)+1} D(\beta) \sum_{j}^{m} \frac{a_j A_j(\alpha) B(\alpha)}{B_j(\alpha)} = s^r t^{m(n-1)} cC(\beta) B(\alpha),$$

which is satisfied identically in the α's and β's if

$s = D(\beta) \sum_{j}^{m} \frac{a_j A_j(\alpha) B(\alpha)}{B_j(\alpha)}$, $t = cC(\beta)B(\alpha)$. Hence a solution of (5:31) is given by (5:32).

Let $x_{ij} = \lambda_{ij}$, $y_j = \mu_j$ be any solution of (5:31). If $\alpha_{ij} = \lambda_{ij}$, $\beta_k = \mu_k$ then $s = t$ and the solution becomes $x_{ij} = \lambda_{ij} t$, $y_k = \mu_k t$, which is equivalent to the given solution provided $t \neq 0$. But if $t = 0$, then $B(\lambda) = \prod_{j}^{m} B_j(\lambda) = 0$ and hence some $B_j(\lambda) = 0$. But if some $B_j(\lambda) = 0$, then $\sum_{i}^{n} \frac{b_{ij}}{\lambda_{ij}} = 0$ since $A_j(\lambda) \neq 0$. This is impossible from (5:31).

If $a_1 = c = 1$, $a_2 = a_3 = \ldots = 0$ (5:31) reduces to[22/]

22/ Leonard E. Dickson, op. cit., pp. 688 - 691.

$$\sum_{i}^{n} \frac{b_{i1}}{x_{i1}} = \sum_{k}^{r} \frac{c_k}{y_k}.$$

Theorem 5:8. The equation[23/]

23/ A. A. Aucoin, "Solution of a Diophantine Equation," Boletin Matematico, 12: 45 - 46 , April, 1939.

$$f(x) = g(y), \tag{5:34}$$

where $f(x) = \prod_{j} \sum_{i}^{n_j} \frac{a_{ij}}{x_{ij}}$, $g(y) = \sum_{j}^{t} \frac{b_j}{y_j}$, all the letters represent integers, and x_{ij} , y_j are unknown, has integral solutions and every solution for which the members do not vanish is equivalent to one of the solutions given by

(5:35) $\qquad x_{ij} = \alpha_{ij} A(\alpha) B(\beta) f(\alpha), \qquad y_j = \beta_j [A(\alpha)B(\beta)]^m \, g(\beta) [f(\alpha)]^{m-1},$

where $A_j(\alpha) = \prod_i^{n_j} \alpha_{ij}$, $A(\alpha) = \prod_j^m A_j(\alpha)$, $B(\beta) = \prod_j^p \beta_j$, and α_{ij}, β_k ($\neq 0$) are arbitrary integers.

Proof. Equation (5:34) may be written in the equivalent form

(5:36) $\qquad A(x)B(y)f(x) = A(x)B(y)g(y).$

If $x_{ij} = \alpha_{ij} t$, $y_j = \beta_j s t^{m-1}$, then (5:36) becomes

$$s^p t^{\sum_j^m n_j + p(m-1) - m} A(\alpha)B(\beta)f(\alpha) = s^{p-1} t^{\sum_j^m n_j + (p-1)(m-1)} A(\alpha)B(\beta)g(\beta),$$

which is satisfied identically in the α's and β's if $s = A(\alpha)B(\beta)g(\beta)$, $t = A(\alpha)B(\beta)f(\alpha)$. Hence a solution of (5:34) is given by (5:35).

If $x_{ij} = \lambda_{ij}$, $y_j = \mu_j$ is a solution of (5:34) and the choice $\alpha_{ij} = \lambda_{ij}$, $\beta_j = \mu_j$, is made, $s = t$ and the solution becomes $x_{ij} = \lambda_{ij} t$, $y_j = \mu_j t^m$ which is equivalent to the given solution provided $t \neq 0$. But since $\lambda_{ij}, \mu_j \neq 0$, $t = A(\lambda)B(\mu)f(\lambda) = 0$ if and only if $f(\lambda) = 0$.

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BIBLIOGRAPHY

A. A. Aucoin, "A Note on a Diophantine Equation", National Mathematics Magazine (To be published later).

A. A. Aucoin, "Diophantine Equations of Degree n", Bulletin of the American Mathematical Society (To be published later).

A. A. Aucoin and W. V. Parker, "Diophantine Equations whose Members are Homogeneous", Bulletin of the American Mathematical Society, XLV: 330-333, April, 1939.

A. A. Aucoin, "Solution of a Diophantine Equation", Boletin Matematico, XII: 45-46, April, 1939.

Robert D. Carmichael, Diophantine Analysis, (New York: John Wiley and Sons, Inc., 1915).

A. Desboves, "Mémoire sur la résolution en nombres entières de l'équation $ax^m + by^m = cz^n$, Nouvelles Annales de Mathématiques, XVIII (2): 481, 1879.

Leonard E. Dickson, History of the Theory of Numbers, (Washington: Carnegie Institute of Washington, 1920), II.

W. V. Parker and A. A. Aucoin, "On Cubic Diophantine Equation", National Mathematics Magazine, XIII: 115-117, December, 1938.

W. V. Parker and A. A. Aucoin, "Solution of a Cubic Diophantine Equation", Tôhoku Mathematical Journal, XLV: 324-328, Part 2, March, 1939.

Volume II of Dickson's History of the Theory of Numbers is a comprehensive listing of all the important results and methods in Diophantine analysis up to its time of publication.

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